

Quantum Steering without Inequalities

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We show that, for any two-qubit state, quantum steering can be proven without testing the violation of steering inequalities. We show that steerability is proven if Bob's normalized conditional states after Alice's measurements are pure. This method, which may be seen as the quantum steering analog of Greenberger-Horne-Zeilinger-like tests of Bell nonlocality without Bell inequalities, offers advantages with respect to the existing methods for experimentally testing quantum steering.

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Introduction.—Quantum steering is a form of quantum nonlocality intermediate between entanglement and Bell nonlocality. The concept of steering goes back to Schrödinger [1] and was properly formalized by Wiseman, Jones, and Doherty [2]. In the steering scenario, Alice prepares a bipartite system, keeps one particle and sends the other one to Bob. She announces that the particle Bob receives is entangled with the one she holds, and thus that she has the ability to “steer” the state of Bob's particle at a distance. This means that she could prepare Bob's particle in different states by measuring her particle using different settings. However, Bob does not trust Alice; Bob worries that she may send him some nonentangled particles and fabricate the results using her knowledge about the local hidden state (LHS) of his particles. Bob's task is to prove that no LHS exists.

At the beginning, Alice announces the possible ensembles $\{E^A : \hat{A}\}$ into which she can steer Bob's states. Here \hat{A} denotes Alice's local measurement, $E^A \equiv \{\tilde{\rho}_a^A : a\}$, with $\tilde{\rho}_a^A$ denoting the state Bob will get if Alice measures her particle with \hat{A} and obtains result a , and the tilde represents that this state is unnormalized (its norm is the probability of Alice gets result a). Then, Bob randomly picks up an ensemble E^A , and asks Alice to prepare it. Then, Alice measures \hat{A} on her system, and tells Bob which ρ_a^A she has steered (at that moment, Alice knows which a she gets, so she knows Bob's state). After many runs, Bob can verify whether each state Alice announced has the correct probability $\text{tr}(\tilde{\rho}_a^A)$.

If Bob's system admits a LHS $\{\wp_\xi \rho_\xi\}$, where ρ_ξ 's are states that Bob does not know (but Alice may know), and $\wp_\xi > 0$ is the probability of ρ_ξ , with $\sum_\xi \wp_\xi = 1$, then Alice could attempt to fabricate her results, using her knowledge of ξ . Specifically, Alice could cheat Bob if there exists an ensemble $\{\wp_\xi \rho_\xi\}$ and a stochastic map $\wp(a|\hat{A}, \xi)$ from ξ to a , such that $\tilde{\rho}_a^A = \sum_\xi \wp(a|\hat{A}, \xi) \wp_\xi \rho_\xi$, with $\sum_a \wp(a|\hat{A}, \xi) = 1$. Conversely, if the correlations

cannot be reproduced in this way, then no LHS exists, and Alice will convince Bob that she can actually steer his state.

Steering is usually proven by the violation of a *steering inequality* [3, 4]. In analogy to a Bell inequality, which is satisfied by any local hidden variable model, a steering inequality is a correlation inequality satisfied by any LHS that Alice could possibly prepare. To test a steering inequality, first Bob asks Alice to perform one of a few possible measurements on her system and to tell him what the measurement result was. Then, Bob performs a measurement on his own system to check whether or not he holds the correct state. Repeating this procedure a sufficient number of times allows Bob to test the violation of the steering inequality. Several steering inequalities have been experimentally tested recently [4–7].

Here we introduce a method to prove steering without using steering inequalities. The idea somehow resembles Greenberger-Horne-Zeilinger (GHZ) [8] and “all-versus-nothing” (AVN) [9, 10] proofs of Bell nonlocality without using Bell inequalities, but in a way goes far beyond. We will prove that, for any two-qubit state, quantum steering can be proven whenever Bob's normalized conditional states (which are those obtained after Alice performs the measurements Bob asked her to perform) are pure. This offers an alternative transparent proof of nonexistence of LHSs. In addition, it provides an effective method to experimentally test quantum steering which has some advantages with respect to the existing methods.

Our main result is the following Theorem.

Theorem: For any two-qubit entangled state ρ_{AB} , Bob can prove quantum steering by checking that his normalized conditional states $\rho_a^{\hat{A}}$ ($a = 0, 1$) are two different pure states.

In the steering scenario, Alice prepares a two-qubit state ρ_{AB} , she keeps one qubit and sends the other to Bob. On her qubit, Alice is asked by Bob to perform a

projective measurement

$$\mathcal{P}_a^{\hat{n}} = [\mathbb{1} + (-1)^a \hat{n} \cdot \vec{\sigma}] / 2, \quad (1)$$

and to tell the measurement result of a . Here $\hat{n} = (n_x, n_y, n_z)$ is the measurement direction, a (with $a = 0, 1$) is Alice's measurement result, $\mathbb{1}$ is the 2×2 identity matrix, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the vector of the Pauli matrices.

After Alice's measurement, Bob's two conditional states $\tilde{\rho}_a^{\hat{n}}$ become

$$\tilde{\rho}_0^{\hat{n}} = \text{tr}(\mathcal{P}_0^{\hat{n}} \rho_{AB}) = \text{tr}(|+\hat{n}\rangle\langle+\hat{n}| \rho_{AB}), \quad (2a)$$

$$\tilde{\rho}_1^{\hat{n}} = \text{tr}(\mathcal{P}_1^{\hat{n}} \rho_{AB}) = \text{tr}(|-\hat{n}\rangle\langle-\hat{n}| \rho_{AB}). \quad (2b)$$

In general, the normalized conditional states $\rho_a^{\hat{n}} = \tilde{\rho}_a^{\hat{n}} / \text{tr}(\tilde{\rho}_a^{\hat{n}})$ are not pure. If both $\rho_a^{\hat{n}}$ ($a = 0, 1$) are pure, then the state ρ_{AB} possesses the following uniform form:

$$\begin{aligned} \rho_{AB} = & \mathcal{P}_0^{\hat{n}} \otimes \tilde{\rho}_0^{\hat{n}} + \mathcal{P}_1^{\hat{n}} \otimes \tilde{\rho}_1^{\hat{n}} + |+\hat{n}\rangle\langle-\hat{n}| \otimes \mathcal{M} \\ & + |-\hat{n}\rangle\langle+\hat{n}| \otimes \mathcal{M}^\dagger, \end{aligned} \quad (3)$$

where \mathcal{M} is a 2×2 complex matrix under the positivity condition of ρ_{AB} , and $\mathcal{M}^\dagger = (\mathcal{M})^\dagger$.

Remark 1: Suppose that Alice prepares a product state $\rho_{AB} = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|$. It can be verified that, for any projective measurement $\mathcal{P}_a^{\hat{n}}$ ($\mathcal{P}_a^{\hat{n}} \neq |\psi_A\rangle\langle\psi_A|$ and $|\psi_A^\perp\rangle\langle\psi_A^\perp|$) performed by Alice, Bob always obtains the same two equally pure normalized conditional states $\rho_0^{\hat{n}} = \rho_1^{\hat{n}} = |\psi_B\rangle\langle\psi_B|$, which means Alice cannot steer Bob's state. Moreover, Bob can obtain two equally pure normalized conditional states if and only if ρ_{AB} is a direct-product state. Hence, hereafter we will assume that $\rho_0^{\hat{n}}$ and $\rho_1^{\hat{n}}$ are two different pure states.

The steerability of ρ_{AB} is invariant under the local unitary transformation $\mathcal{U}_A \otimes \mathbb{1}$. In other words, the steerability of ρ_{AB} is equivalent to that of the *rotated* state $\varrho_{AB} = (\mathcal{U}_A \otimes \mathbb{1})\rho_{AB}(\mathcal{U}_A^\dagger \otimes \mathbb{1})$. It is always possible for Alice to choose an appropriate unitary matrix \mathcal{U} that rotates the direction \hat{n} to the direction \hat{z} . Therefore, without loss of generality, we can initially set $\hat{n} = \hat{z}$ by studying the rotated state ϱ_{AB} instead of ρ_{AB} .

Let us denote the rotated density matrix ϱ_{AB} as

$$\varrho_{AB} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}. \quad (4)$$

After Alice performs a projective measurement in the \hat{z} -direction, Bob's unnormalized conditional states are

$$\tilde{\rho}_0^{\hat{z}} = \text{tr}_A[|0\rangle\langle 0| \otimes \mathbb{1}] \varrho_{AB} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad (5a)$$

$$\tilde{\rho}_1^{\hat{z}} = \text{tr}_A[|1\rangle\langle 1| \otimes \mathbb{1}] \varrho_{AB} = \begin{pmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{pmatrix}. \quad (5b)$$

The corresponding pure conditional states are denoted as

$$\rho_0^{\hat{z}} = |\varphi_1\rangle\langle\varphi_1| = \frac{1}{\mu_1} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad (6a)$$

$$\rho_1^{\hat{z}} = |\varphi_2\rangle\langle\varphi_2| = \frac{1}{\mu_2} \begin{pmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{pmatrix}, \quad (6b)$$

with $\mu_1 = \text{tr}(\tilde{\rho}_0^{\hat{z}}) = c_{11} + c_{22}$ and $\mu_2 = \text{tr}(\tilde{\rho}_1^{\hat{z}}) = c_{33} + c_{44}$. Then, the expression for ϱ_{AB} is

$$\begin{aligned} \varrho_{AB} = & \mu_1 |0\rangle\langle 0| \otimes |\varphi_1\rangle\langle\varphi_1| + \mu_2 |1\rangle\langle 1| \otimes |\varphi_2\rangle\langle\varphi_2| \\ & + |0\rangle\langle 1| \otimes \mathcal{M} + |1\rangle\langle 0| \otimes \mathcal{M}^\dagger, \end{aligned} \quad (7)$$

with

$$\mathcal{M} = \begin{pmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{pmatrix}. \quad (8)$$

Let $\rho_0^{\hat{z}} \neq \rho_1^{\hat{z}}$. To prove the Theorem, let us first prove two Lemmas.

Lemma 1: $\mathcal{M} = \mathbf{0}$ if and only if ϱ_{AB} is separable.

Proof: From Eq. (7), it is easy to see that $\mathcal{M} = \mathbf{0}$ implies that ϱ_{AB} is separable. To prove the converse, one needs the definition of separability:

$$\varrho_{AB} = \sum_i p_i \tau_{Ai} \otimes \tau_{Bi}, \quad (9)$$

where τ_{Ai} and τ_{Bi} are, respectively, Alice and Bob's local density matrices, and $p_i > 0$ satisfy $\sum_i p_i = 1$. For convenience, let τ_{Ai}^{mn} ($m, n = 1, 2$) denote the element of Alice's density matrix τ_{Ai} . By calculating $\text{tr}(|0\rangle\langle 0| \varrho_{AB})$ and $\text{tr}(|1\rangle\langle 1| \varrho_{AB})$, one has, from Eqs. (7) and (9), that

$$\sum_i p_i \tau_{Ai}^{11} \tau_{Bi} = \mu_1 |\varphi_1\rangle\langle\varphi_1|, \quad (10a)$$

$$\sum_i p_i \tau_{Ai}^{22} \tau_{Bi} = \mu_2 |\varphi_2\rangle\langle\varphi_2|. \quad (10b)$$

Let $|\varphi_1^\perp\rangle$ and $|\varphi_2^\perp\rangle$ be two pure states that are orthogonal to $|\varphi_1\rangle$ and $|\varphi_2\rangle$, respectively. Explicitly,

$$|\varphi_1^\perp\rangle\langle\varphi_1^\perp| = \frac{1}{\mu_1} \begin{pmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{pmatrix}, \quad (11a)$$

$$|\varphi_2^\perp\rangle\langle\varphi_2^\perp| = \frac{1}{\mu_2} \begin{pmatrix} c_{44} & -c_{34} \\ -c_{43} & c_{33} \end{pmatrix}. \quad (11b)$$

Notice that

$$\text{tr}[\text{Eq. (10a)} \times |\varphi_1^\perp\rangle\langle\varphi_1^\perp|] = 0, \quad (12a)$$

$$\text{tr}[\text{Eq. (10b)} \times |\varphi_2^\perp\rangle\langle\varphi_2^\perp|] = 0. \quad (12b)$$

Thus, for any index i , we have

$$\tau_{Ai}^{11} \text{tr}(\tau_{Bi} |\varphi_1^\perp\rangle\langle\varphi_1^\perp|) = 0, \quad (13a)$$

$$\tau_{Ai}^{22} \text{tr}(\tau_{Bi} |\varphi_2^\perp\rangle\langle\varphi_2^\perp|) = 0, \quad (13b)$$

which results in

$$\tau_{Ai}^{11} \tau_{Ai}^{22} [\text{tr}(\tau_{Bi} |\varphi_1^\perp\rangle\langle\varphi_1^\perp|) + \text{tr}(\tau_{Bi} |\varphi_2^\perp\rangle\langle\varphi_2^\perp|)] = 0. \quad (14)$$

Since $|\varphi_1^\perp\rangle$ and $|\varphi_2^\perp\rangle$ are two different pure states that cannot be simultaneously perpendicular to the state τ_{Bi} , thus the only possibility is $\tau_{Ai}^{11}\tau_{Ai}^{22} = 0$, which yields $\tau_{Ai}^{12} = \tau_{Ai}^{21} = 0$ due to the positivity condition of density matrix τ_{Ai} . Therefore,

$$\mathcal{M} = \text{tr}(|1\rangle\langle 0| \varrho_{AB}) = \sum_i p_i \tau_{Ai}^{12} \tau_{Bi} = \mathbf{0}. \quad (15)$$

Lemma 1 is henceforth proved. \blacksquare

Lemma 2: The state ϱ_{AB} admits a LHS (which means that it is not steerable) if and only if $\mathcal{M} = \mathbf{0}$.

Proof: $\mathcal{M} = \mathbf{0}$ implies ϱ_{AB} is separable, thus ϱ_{AB} admits a LHS. Now we focus on the proof of necessity. Suppose ϱ_{AB} has a LHS, then there exists an ensemble $\{\wp_\xi \rho_\xi\}$ with $\sum_\xi \wp_\xi = 1$, $\wp_\xi > 0$ (note that $\wp_\xi = 0$ is excluded, since it means that ρ_ξ does not appear in the ensemble), and a stochastic map $\wp(a|\hat{A}, \xi)$ satisfying $\tilde{\rho}_a^{\hat{A}} = \sum_\xi \wp(a|\hat{A}, \xi) \wp_\xi \rho_\xi$, with $\sum_a \wp(a|\hat{A}, \xi) = 1$, $\wp(a|\hat{A}, \xi) \geq 0$. Here \hat{A} is Alice's measurement [2].

If Alice's measurement setting is $\{\hat{z}, \hat{x}\}$, then we have

$$\tilde{\rho}_0^{\hat{z}} = \mu_1 |\varphi_1\rangle\langle\varphi_1| = \sum_\xi \wp(0|\hat{z}, \xi) \wp_\xi \rho_\xi, \quad (16a)$$

$$\tilde{\rho}_1^{\hat{z}} = \mu_2 |\varphi_2\rangle\langle\varphi_2| = \sum_\xi \wp(1|\hat{z}, \xi) \wp_\xi \rho_\xi, \quad (16b)$$

$$\begin{aligned} \tilde{\rho}_0^{\hat{x}} &= \frac{1}{2}(\mu_1 |\varphi_1\rangle\langle\varphi_1| + \mu_2 |\varphi_2\rangle\langle\varphi_2| + \mathcal{M} + \mathcal{M}^\dagger) \\ &= \sum_\xi \wp(0|\hat{x}, \xi) \wp_\xi \rho_\xi, \end{aligned} \quad (16c)$$

$$\begin{aligned} \tilde{\rho}_1^{\hat{x}} &= \frac{1}{2}(\mu_1 |\varphi_1\rangle\langle\varphi_1| + \mu_2 |\varphi_2\rangle\langle\varphi_2| - \mathcal{M} - \mathcal{M}^\dagger) \\ &= \sum_\xi \wp(1|\hat{x}, \xi) \wp_\xi \rho_\xi. \end{aligned} \quad (16d)$$

Because $\langle\varphi_1^\perp|\mathbf{Eq. (16a)}|\varphi_1^\perp\rangle = 0$ and $\langle\varphi_2^\perp|\mathbf{Eq. (16b)}|\varphi_2^\perp\rangle = 0$, then, for any index ξ , we have

$$\langle\varphi_1^\perp|\rho_\xi|\varphi_1^\perp\rangle \wp(0|\hat{z}, \xi) = 0, \quad (17a)$$

$$\langle\varphi_2^\perp|\rho_\xi|\varphi_2^\perp\rangle \wp(1|\hat{z}, \xi) = 0. \quad (17b)$$

From **Eq. (17a)** $\times \langle\varphi_2^\perp|\rho_\xi|\varphi_2^\perp\rangle + \mathbf{Eq. (17b)}$ $\times \langle\varphi_1^\perp|\rho_\xi|\varphi_1^\perp\rangle$, we have

$$\langle\varphi_1^\perp|\rho_\xi|\varphi_1^\perp\rangle \langle\varphi_2^\perp|\rho_\xi|\varphi_2^\perp\rangle = 0, \quad (18)$$

which implies that $\rho_\xi \in \{|\varphi_1\rangle\langle\varphi_1|, |\varphi_2\rangle\langle\varphi_2|\}$ for any ξ . Combining this result with **Eq. (16c)**, one finds that **Eq. (16c)** is valid only if $\mathcal{M} + \mathcal{M}^\dagger$ is a linear combination of $|\varphi_1\rangle\langle\varphi_1|$ and $|\varphi_2\rangle\langle\varphi_2|$ (let us denote it as $\mathcal{M} + \mathcal{M}^\dagger = (\alpha_x |\varphi_1\rangle\langle\varphi_1| + \beta_x |\varphi_2\rangle\langle\varphi_2|)/2$, with $\alpha_x, \beta_x \in \mathbb{R}$). Otherwise, no LHS can produce Bob's conditional states.

Similarly, if Alice's measurement setting is $\{\hat{z}, \hat{y}\}$, then

$$\begin{aligned} \tilde{\rho}_0^{\hat{y}} &= \frac{1}{2}[\mu_1 |\varphi_1\rangle\langle\varphi_1| + \mu_2 |\varphi_2\rangle\langle\varphi_2| - i(\mathcal{M} - \mathcal{M}^\dagger)] \\ &= \sum_\xi \wp(0|\hat{y}, \xi) \wp_\xi \rho_\xi. \end{aligned} \quad (19)$$

Following the same analysis as above, one finds that **Eq. (19)** is valid only if $\mathcal{M} - \mathcal{M}^\dagger = i(\alpha_y |\varphi_1\rangle\langle\varphi_1| + \beta_y |\varphi_2\rangle\langle\varphi_2|)/2$, with α_y, β_y the real numbers. Thus if there exists a LHS for the state ϱ_{AB} , then \mathcal{M} must be the form $\mathcal{M} = \alpha |\varphi_1\rangle\langle\varphi_1| + \beta |\varphi_2\rangle\langle\varphi_2|$, with $\alpha = \alpha_x + i\alpha_y$, $\beta = \beta_x + i\beta_y$. Substituting \mathcal{M} in **Eq. (7)**, we have

$$\begin{aligned} \varrho_{AB} &= \mu_1 \begin{pmatrix} 1 & \alpha \\ \alpha^* & 0 \end{pmatrix} \otimes |\varphi_1\rangle\langle\varphi_1| \\ &+ \mu_2 \begin{pmatrix} 0 & \beta \\ \beta^* & 1 \end{pmatrix} \otimes |\varphi_2\rangle\langle\varphi_2|. \end{aligned} \quad (20)$$

Now we construct the following two projectors:

$$\mathcal{Q}_1 = |\chi_1\rangle\langle\chi_1| \otimes |\varphi_2^\perp\rangle\langle\varphi_2^\perp|, \quad (21a)$$

$$\mathcal{Q}_2 = |\chi_2\rangle\langle\chi_2| \otimes |\varphi_1^\perp\rangle\langle\varphi_1^\perp|, \quad (21b)$$

where $|\chi_1\rangle$ is the eigenvector of $\begin{pmatrix} 1 & \alpha \\ \alpha^* & 0 \end{pmatrix}$ with eigenvalue $v_1 = \frac{1}{2}(1 - \sqrt{1 + 4|\alpha|^2}) \leq 0$, and $|\chi_2\rangle$ is the eigenvector of $\begin{pmatrix} 0 & \beta \\ \beta^* & 1 \end{pmatrix}$ with eigenvalue $v_2 = \frac{1}{2}(1 - \sqrt{1 + 4|\beta|^2}) \leq 0$. Because ϱ_{AB} is a density matrix, then one has

$$\text{tr}(\varrho_{AB} \mathcal{Q}_1) = v_1 \mu_1 |\langle\varphi_2^\perp|\varphi_1\rangle|^2 \geq 0, \quad (22a)$$

$$\text{tr}(\varrho_{AB} \mathcal{Q}_2) = v_2 \mu_2 |\langle\varphi_1^\perp|\varphi_2\rangle|^2 \geq 0. \quad (22b)$$

Thus $\alpha = \beta = 0$ and $\mathcal{M} = \mathbf{0}$. Lemma 2 is henceforth proved. \blacksquare

Remark 2: Three measurement settings were used in the proof of Lemma 2. This does not mean that we need a three-setting protocol to demonstrate steering. For a given entangled state ϱ_{AB} , Lemma 2 shows that $\mathcal{M} + \mathcal{M}^\dagger$ and $\mathcal{M} - \mathcal{M}^\dagger$ cannot be linearly expanded of $|\varphi_1\rangle\langle\varphi_1|$ and $|\varphi_2\rangle\langle\varphi_2|$ simultaneously (because that means $\mathcal{M} = \mathbf{0}$ and ρ_{AB} is separable). Thus, for a given entangled state ϱ_{AB} , if $\mathcal{M} + \mathcal{M}^\dagger \neq (\alpha_x |\varphi_1\rangle\langle\varphi_1| + \beta_x |\varphi_2\rangle\langle\varphi_2|)/2$, then the two-setting protocol using $\{\hat{z}, \hat{x}\}$ is sufficient to demonstrate quantum steering; otherwise, the protocol using $\{\hat{z}, \hat{y}\}$ will work.

We are now ready to prove the main Theorem.

Proof of the Theorem: The proof is based on Lemmas 1 and 2. For a given two-qubit state of the form (3), after Alice performs the projective measurements, Bob obtains two different pure normalized conditional states. Then, the following three propositions are equivalent: (i) $\mathcal{M} \neq \mathbf{0}$. (ii) ρ_{AB} is entangled. (iii) No LHS exists for ρ_{AB} , thus ρ_{AB} is steerable. The Theorem is henceforth proved. \blacksquare

Steering without inequalities vs steering inequalities.— Here we discuss how the method for detecting steering introduced in this Letter compares with previous steering inequalities. For this purpose, consider the following

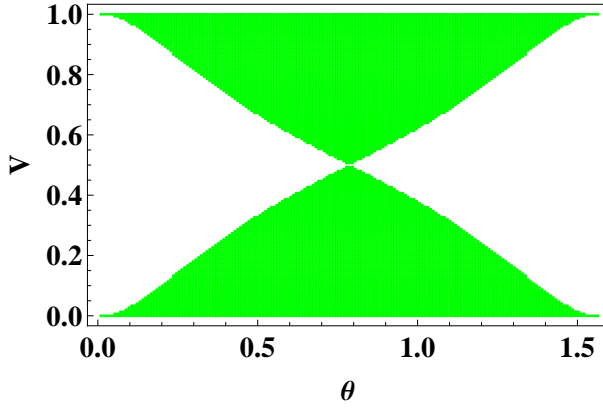


FIG. 1: (Color online) Detecting steering of state (23) using the inequality (24). Steerability can be detected using the inequality (24) only in the green region. In contrast, according to the proof of steering without inequalities, the state (23) is steerable for the whole region in which it is entangled.

entangled state:

$$\rho_X = \begin{pmatrix} Vc_\theta^2 & 0 & 0 & Vs_\theta c_\theta \\ 0 & (1-V)s_\theta^2 & (1-V)s_\theta c_\theta & 0 \\ 0 & (1-V)s_\theta c_\theta & (1-V)c_\theta^2 & 0 \\ Vs_\theta c_\theta & 0 & 0 & Vs_\theta^2 \end{pmatrix}, \quad (23)$$

with $V \in [0, 1/2) \cup (1/2, 1]$, $\theta \in (0, \pi/2)$ (otherwise the state is not entangled), $c_\theta = \cos\theta$, and $s_\theta = \sin\theta$. It can be verified that, after Alice performs the projective measurement along the \hat{x} -direction, Bob's conditional states are $\tilde{\rho}_0^{\hat{x}} = \frac{1}{2}|\psi_1\rangle\langle\psi_1|$ and $\tilde{\rho}_1^{\hat{x}} = \frac{1}{2}|\psi_2\rangle\langle\psi_2|$, with $|\psi_1\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$ and $|\psi_2\rangle = \cos\theta|0\rangle - \sin\theta|1\rangle$. Quantum steering of the state (23) can be demonstrated by the two-setting protocol using $\{\hat{x}, \hat{z}\}$.

Alternatively, to detect steering, one can consider the following class of N -setting inequality introduced in Ref. [4] to detect steering of two-qubit states:

$$\mathcal{S}_N = \frac{1}{N} \sum_{k=1}^N \langle A_k \vec{\sigma}_k^B \rangle - C_N \leq 0. \quad (24)$$

By checking a 10-setting steering inequality of the form (24), we observe that, for some regions of V and θ , the steering inequality cannot detect steering (see Fig. 1).

In contrast, steering without inequalities will always allow us to detect steering for any state of the form (23). In this sense, steering without inequalities is different from Bell nonlocality without inequalities. While Bell nonlocality proofs (GHZ and AVN) applies only to a smaller set of quantum states [12–14] than Bell inequalities, the previous example shows that steering without inequalities applies to two-qubit states for which steering inequalities fail.

Conclusions.—We have demonstrated quantum steering without inequalities for any two-qubit entangled

state. The proof offers a transparent argument for the nonexistence of LHSs without resorting to steering inequalities. In contrast with the connection between proofs with and without inequalities of Bell nonlocality (where GHZ and AVN proofs are not valid for two-qubit states, apply to a smaller set of quantum states than Bell inequalities, and are not particularly useful to experimentally test Bell nonlocality), steering without inequalities is valid for two-qubit systems, applies to more two-qubit states than the steering inequalities of Ref. [4], and offers an effective experimental method to detect steering based on testing the purity (for instance, through state tomography) of Bob's reduced state. This provides a simple alternative to the existing experimental method for detecting steering [4–7]. We expect further developments along these lines in the near future.

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